Astronomy 218
Modeling Stars
Modeling stellar structure relies on 4 vital physical concepts, each expressed as an equation.

$$\frac{dM}{dr} = 4\pi r^2 \rho(r) \quad \frac{dL}{dr} = 4\pi r^2 \rho(r)\varepsilon(r) \quad \frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

**Energy Transport**

The form of the transport equation depends on the dominant energy transport process.

- Radiative
  \[
  \frac{dT}{dr} = -\frac{3\rho(r)\kappa(r)L(r)}{64\pi\sigma_{SB}T(r)^3r^2}
  \]

- Convective
  \[
  \frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right)\frac{T(r)}{P(r)}\frac{dP}{dr}
  \]

The solution to this system of equations is $T(r)$, $\rho(r)$ and $L(r)$, along with derived quantities like $P(r)$ and $\mu(r)$.
Russell’s publication of the original HR diagram in 1913, followed rapidly by analysis of star clusters, generated a lot of excitement in the theory of stellar structure. Rapid advance was made, using simplified physics. For example, using a polytropic equation of state ($P = C \rho^n$), the equations of stellar structure reduce to the Lane-Emden equations, for which solutions were tabulated. By 1926, when Eddington’s *The Internal Constitution of the Stars* was published, the basic picture of solar structure, including nuclear power, had emerged.
Hidden in the equations of stellar structure is a wealth of necessary atomic and nuclear data.

$\kappa(r)$ is a function of atomic cross sections needed to calculate $\kappa(\rho,T,\mu,X,Y,Z,\ldots)$, measured in terrestrial laboratories, and the calculated $T(r)$, $\rho(r)$, ...

$\varepsilon(r)$ is determined by nuclear cross sections, measured with terrestrial accelerators, which are combined to $\varepsilon(\rho,T,X,Y,Z,\ldots)$ and convolved with the calculated $T(r)$, $\rho(r)$, ...

Another important input is the Equation of State $P(\rho,T,\mu,X,Y,Z,\ldots)$, including contributions from atoms, ions, electrons and photons whose relative importance are determined by $T(r)$, $\rho(r)$, ...
For very simple expressions for the Equation of state, $P(\rho, T, \mu, X, Y, Z, \ldots)$, opacity, $\kappa(\rho, T, \mu, X, Y, Z, \ldots)$, and energy generation rate, $\varepsilon(\rho, T, X, Y, Z, \ldots)$, it is possible to solve the equations of stellar structure directly.

However, including the full detail in these important physical elements renders the equations of stellar structure too complex to solve analytically.

Instead, they are solved numerically.

The results, the standard solar model and models for other stars, are among the first triumphs of computational physics.
Solar Solution

Solving the equations of stellar structure for an individual star requires setting the **boundary conditions** to reflect that star.

For example, for the Sun we set $M(0) = 0$, $L(0) = 0$, & $L(R_{\odot}) = L_{\odot}$, $M(R_{\odot}) = M_{\odot}$, along with solar values of the composition ($X$, $Y$, $Z$).

The **solution** of the equations of stellar structure reveals distributions of temperature and especially density which are strongly **centrally peaked**.
Solution also includes the composition, which reveals conversion of $\text{H} \Rightarrow \text{He}$ (and $\text{C} \& \text{O}$ to $\text{N}$) are restricted to the central 10-20% of the Sun.
Solar Convection

The high temperature of the solar interior fully ionizes the gas, however the decline in $T(r)$ allows atoms to exist in the outer regions.

This divides the solar interior into 2 zones, an inner radiative zone and an outer zone where atoms can exist, increasing the opacity $\kappa(r)$, making convection the most rapid form of energy transport.
Convection in Stars

The more general question of where convection in occurs in stars has several answers.

Radiative\[ \frac{dT}{dr} = -\frac{3\rho(r)\kappa(r)L(r)}{64\pi\sigma_{SB}T(r)^3r^2} \]

Convective\[ \frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right)\frac{T(r)}{P(r)} \frac{dP}{dr} \]

The radiative gradient can exceed the adiabatic gradient when it is large or when the adiabatic gradient is small.

In the Sun, convection occurs when large \( \kappa(r) \) makes the radiative gradient large.

Convection also occurs when \( \varepsilon(T) \) is strongly temperature dependent, making the radiative gradient large.

Ionization causes \( \gamma \rightarrow 1 \), making the adiabatic gradient small.
Our ability to compare the solar model to the sun is limited by the extreme optical depths.

The vast majority of our observations come from the solar surface, not the interior.

Only physical features that influence a star’s photosphere are visible.

For the sun we also have 2 clues from deep inside, helioseismology and neutrinos.
The appearance of granulation provides support for the solar model.

Three-dimensional models of the convective zone replicate the scale of the observed convective cells.
Helioseimology

Additional support for the solar model comes from *Helioseismology*.

Doppler shifts of solar spectral lines indicate a complex pattern of *vibrations* on the surface of the Sun, the result of standing *sound waves* in the solar interior.

Different sound waves probe different depths in the Sun.

As the speed of sound depends on the *temperature*, *density* & *composition* of matter, one can deduce these quantities as a function of radius in the Sun.
Several of the nuclear reactions that are part of the PP chain and CNO cycle are $\beta$ decays, involving the conversion of a proton into a neutron. These weak reactions result in the emission of positrons and neutrinos, $p \rightarrow n + e^+ + \nu$.

The positrons rapidly annihilate with a nearby electron, but the neutrinos stream from the core of the Sun and escape, interacting with virtually nothing.

The flux of solar neutrinos at the Earth is $6 \times 10^{14}$ neutrinos m$^{-2}$ s$^{-1}$, 6 trillion through your hand each sec.

Being able to observe these neutrinos would give us a direct picture of what is happening in the core of the Sun.
Solar Neutrinos Observations

Observing neutrinos is very challenging because neutrinos are no more likely to interact with terrestrial detectors than they are in the Sun.

Huge detector volumes and the ability to observe single interaction events is required.
Neutrino Spectra

More important than imaging the Sun’s thermonuclear core is the information provided by the neutrino spectra.
Detection of solar neutrinos has been ongoing for more than 30 years now. Davis and Koshiba won the 2002 Nobel Prize for their detection.

However, there has always been a deficit in the number of electron neutrinos expected to be emitted by the Sun.

This Solar Neutrino problem was ultimately shown to be the result neutrino oscillations, which interchange the 3 flavors of neutrinos during their passage from the Sun’s core to the Earth, indicating neutrinos have mass.

With oscillations accounted for, we now find that the measured solar neutrino emission is a good match to the predictions of the standard solar model.
Aside from matching observations of solar neutrinos and helioseismology, stellar models need to reproduce the characteristics of the observed stellar distribution.

Here stellar models reproduce the main sequence in the globular cluster NGC6397.

Stellar modeling reveals the importance of metallicity $Z$. 

**Main Sequence Modeling**

$\text{NGC}6397 = 0.004 Z_\odot = 0.012 Z_\odot = 0.027 Z_\odot = 0.045 Z_\odot$
The mass-radius relation for main sequence stars is an empirical result. The fitted formulae quantify the relation, but do not explain its physics.

\[ \frac{R}{R_\odot} = 1.06 \left( \frac{M}{M_\odot} \right)^{0.945} \text{ for } M < 1.66 \, M_\odot \]
\[ \frac{R}{R_\odot} = 1.33 \left( \frac{M}{M_\odot} \right)^{0.555} \text{ for } M > 1.66 \, M_\odot \]

Stellar modeling reveals that the transition coincides with the dominance of CNO cycle burning in the core.

\[ \epsilon \propto T^{20} \] for CNO results in steep \( T \) gradients and a convective core.
Stellar modeling also reveals the physics of the mass-luminosity relation.

For large $M$, radiation pressure plays an increasing role. As $M$ declines below $1 \, M_\odot$, the size of the convective zone increases, encompassing the core around $0.3 \, M_\odot$. 

![Mass-Luminosity Relation](image)

$L/L_\odot \approx 1.3 \times 10^5 \, (M/M_\odot)$ for $M > 20 \, M_\odot$

$L/L_\odot = 0.35 \, (M/M_\odot)^2.62$ for $M < 0.7 \, M_\odot$

$L/L_\odot = 1.02 \, (M/M_\odot)^3.92$ for $M > 0.7 \, M_\odot$
For **massive stars**, convection occurs in the core due to CNO cycle. For **low mass stars**, it occurs near the surface where $\kappa(r)$ grows because of a lack of ionization.
Stellar Seismology

While observations of neutrinos remains solely a solar province, improving technology is allowing helioseismology to be extended to other stars. **Astroseismology** is starting to reveal the structure of more distant stars.

A number of space-based telescopes are providing data, notably the Kepler & COROT missions.
Next Time

Interstellar Gas & Dust

Turn in Homework #3.

Homework #4 will be due 2/11, we’ll go over it in class that day.

Exam #1 will occur on 2/15, covering chapters 13-16.